# Ecuación de Karman

Ecuación de continuidad : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ ECdM$_x$ : $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$

Buscamos la forma conservativa de la ECdM$_x$ :

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} &= \frac{1}{2}\frac{\partial}{\partial x}(u^2) \\ v\frac{\partial u}{\partial y} &= \frac{\partial(u\cdot v)}{\partial y}-\underbrace{u\frac{\partial v}{\partial y}}_{\text{CONTINUIDAD}}=\frac{\partial(u\cdot v)}{\partial y}+u\frac{\partial u}{\partial x}=\frac{\partial(u\cdot v)}{\partial y}+\frac{1}{2}\frac{\partial}{\partial x}(u^2)\end{aligned}\right\}\longrightarrow$$

$$\longrightarrow\quad \frac{1}{2}\frac{\partial}{\partial x}(u^2)+\frac{\partial(u\cdot v)}{\partial y}+\frac{1}{2}\frac{\partial}{\partial x}(u^2)=u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(u\cdot u)+\frac{\partial}{\partial y}(u\cdot v)=u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$$

Le restamos la ecuación de continuidad multiplicada por $u_e$ :

$$\frac{\partial}{\partial x}(u\cdot u)+\frac{\partial}{\partial y}(u\cdot v)=u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$$

$$u_e\frac{\partial u}{\partial x}+u_e\frac{\partial v}{\partial y}=0 \qquad \ominus$$

$$\frac{\partial}{\partial x}(u\cdot u)-\boxed{u_e}\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}(u\cdot v)-\boxed{u_e}\frac{\partial v}{\partial y}=u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$$

$$\downarrow \quad u_e\frac{\partial u}{\partial x}=\frac{\partial(u\cdot u_e)}{\partial x}-u\frac{du_e}{dx}$$

$$\frac{\partial}{\partial x}(u\cdot u)-\frac{\partial(u\cdot u_e)}{\partial x}+u\frac{du_e}{dx}+\frac{\partial}{\partial y}(u\cdot v)-\frac{\partial}{\partial y}(u_e\cdot v)-u_e\frac{du_e}{dx}=\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)$$

Obtenemos :

$$\boxed{\frac{\partial}{\partial x}\Big[u(u-u_e)\Big]+\frac{\partial}{\partial y}\Big[v(u-u_e)\Big]+(u-u_e)\frac{du_e}{dx}=\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)}$$

ECUACIÓN DE TRANSPORTE PARA LA MAGNITUD "$u-u_e$", QUE ES EL DEFECTO DE VELOCIDADES EN LA CAPA LÍMITE

Integramos en la dirección transversal entre 0 y $\infty$ :

$$\frac{d}{dx}\int_0^\infty u\,(u-u_e)\,dy + v\,(u-u_e)\Big|_0^\infty + \frac{du_e}{dx}\int_0^\infty (u-u_e)\,dy = \nu\,\frac{\partial u}{\partial y}\Big|_0^\infty$$

$$\frac{d}{dx}\int_0^\infty \frac{u}{u_e}\,u_e^2\left(\frac{u}{u_e}-1\right)dy - v_p\,(-u_e) + \frac{du_e}{dx}\int_0^\infty u_e\left(\frac{u}{u_e}-1\right)dy = -\,\nu\,\underbrace{\frac{\partial u}{\partial y}\Big|_0}_{\frac{\tau_p}{\mu}}$$

Multiplicando por $-1$ :

$$\frac{d}{dx}\int_0^\infty \frac{u}{u_e}\,u_e^2\left(1-\frac{u}{u_e}\right)dy - v_p\,u_e + \frac{du_e}{dx}\int_0^\infty u_e\left(1-\frac{u}{u_e}\right)dy = \frac{\cancel{\mu}}{\rho}\,\frac{\tau_p}{\cancel{\mu}}$$

Si recordamos las definiciones de los espesores de desplazamiento $\delta_1$ y de cantidad de movimiento $\delta_2$ :

$$\delta_1 = \int_0^\infty \left(1-\frac{\rho u}{\rho_e u_e}\right)dy \qquad\qquad \delta_2 = \int_0^\infty \frac{\rho u}{\rho_e u_e}\left(1-\frac{u}{u_e}\right)dy$$

Y considerando capa límite incompresible ( $\rho$ = cte ) :

$$\delta_1 = \int_0^\infty \left(1-\frac{u}{u_e}\right)dy \qquad\qquad \delta_2 = \int_0^\infty \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy$$

Obtenemos :

$$\frac{d}{dx}\underbrace{\int_0^\infty \frac{u}{u_e}\,u_e^2\left(1-\frac{u}{u_e}\right)dy}_{u_e^2\,\delta_2} - v_p\,u_e + \frac{du_e}{dx}\underbrace{\int_0^\infty u_e\left(1-\frac{u}{u_e}\right)dy}_{u_e\,\delta_1} = \frac{\tau_p}{\rho}$$

$$\frac{d}{dx}\left(u_e^2\,\delta_2\right) - v_p\,u_e + u_e\,\frac{du_e}{dx}\,\delta_1 = \frac{\tau_p}{\rho}$$

$$u_e^2\,\frac{d\delta_2}{dx} + 2\,u_e\,\delta_2\,\frac{du_e}{dx} - v_p\,u_e + u_e\,\frac{du_e}{dx}\,\delta_1 = \frac{\tau_p}{\rho}$$

$\rightarrow H_{12}\,\delta_2$ (lo queremos todo en función de $\delta_2$)

Dividiendo por $u_e^2$ :

$$\frac{d\delta_2}{dx} + 2\,\frac{1}{u_e}\,\frac{du_e}{dx}\,\delta_2 + H_{12}\,\frac{1}{u_e}\,\frac{du_e}{dx}\,\delta_2 = \underbrace{\frac{\tau_p}{\rho\,u_e^2}}_{\frac{C_f}{2}} + \frac{v_p}{u_e}$$

$\frac{C_f}{2}$ , donde $C_f$ es el coeficiente de fricción

Finalmente llegamos a :

$$\boxed{\frac{d\delta_2}{\delta x} + \underbrace{(2 + H_{12})\frac{1}{u_e}\frac{du_e}{dx}\delta_2}_{\text{GRADIENTE DE PRESIONES}} = \underbrace{\frac{C_f}{2}}_{\text{FRICCIÓN CON PARED}} + \underbrace{\frac{v_p}{u_e}}_{\text{SUCCIÓN/SOPLADO}}}$$

## ECUACIÓN DE KARMAN RÉGIMEN INCOMPRESIBLE

Le añadimos la condición inicial $x = x_0 : \delta_2 = \delta_{20}$

**ACCIONES QUE ENGORDAN LA C.L.** $\uparrow \delta_2 \left(\frac{d\delta_2}{dx} > 0\right)$

| SOPLADO | FRICCIÓN | GRADIENTE ADVERSO DE PRESIÓN |
|---|---|---|
| $(v_p > 0)$ | $(C_f > 0)$ | $\left(\frac{du_e}{dx} < 0\right)$ |

$\delta_2$ ESTÁ MUY RELACIONADO CON LA RESISTENCIA AERODINÁMICA, POR LO QUE ES MUY RECOMENDABLE ESTUDIAR CÓMO EVOLUCIONA $\delta_2$.